

Dynamic Portfolio Optimisation

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What questions does dynamic optimisation address?

1. Assume a manager has two strategies:

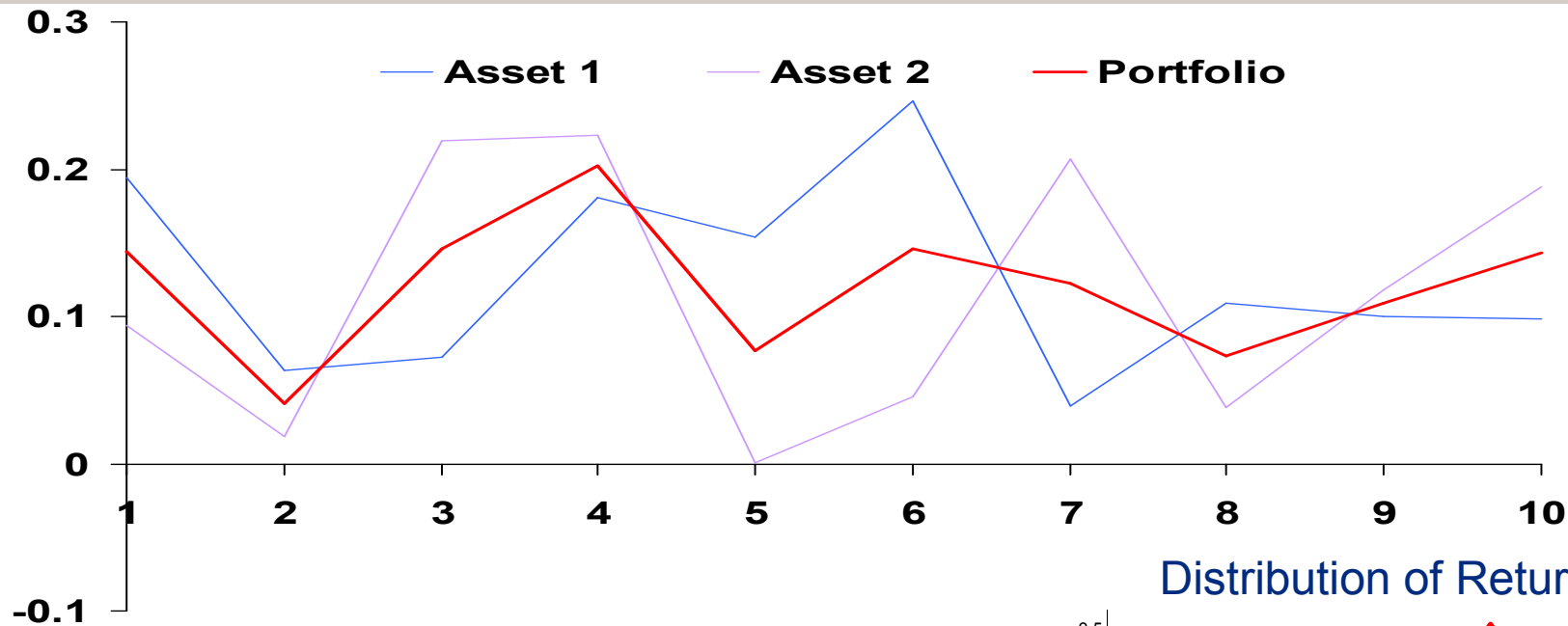
1. The first strategy has a high return but decays quickly.
2. The second has a lower expected return but decays more slowly.

How should the manager use this information in constructing his portfolio?

Take into account transaction costs and problem becomes interesting!

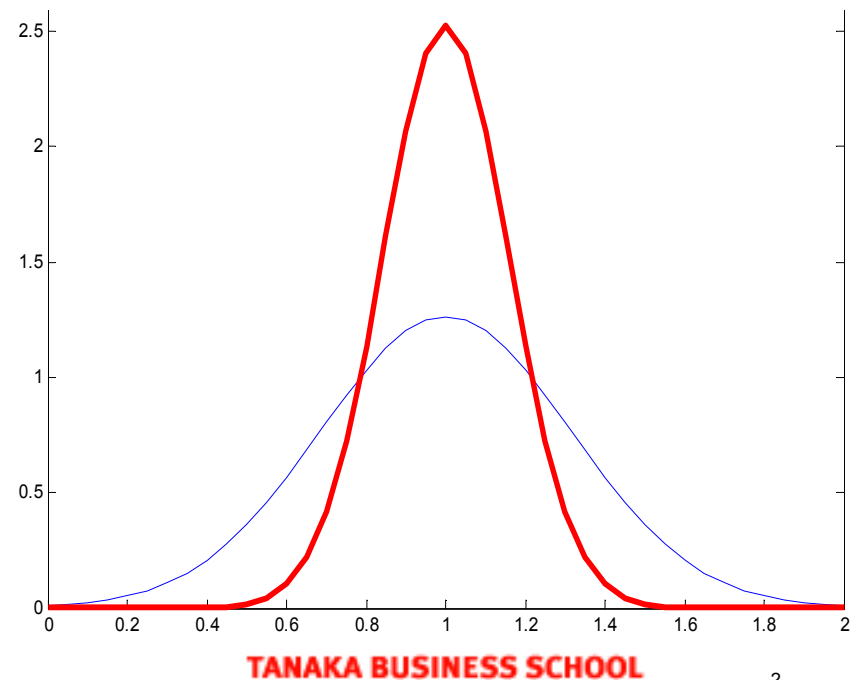
2. Multiperiod optimisation introduces hedging motives for holding some assets. These asset hedge against future falls in expected returns in the strategies (or the future investment opportunity set).

Diversification of within period risk

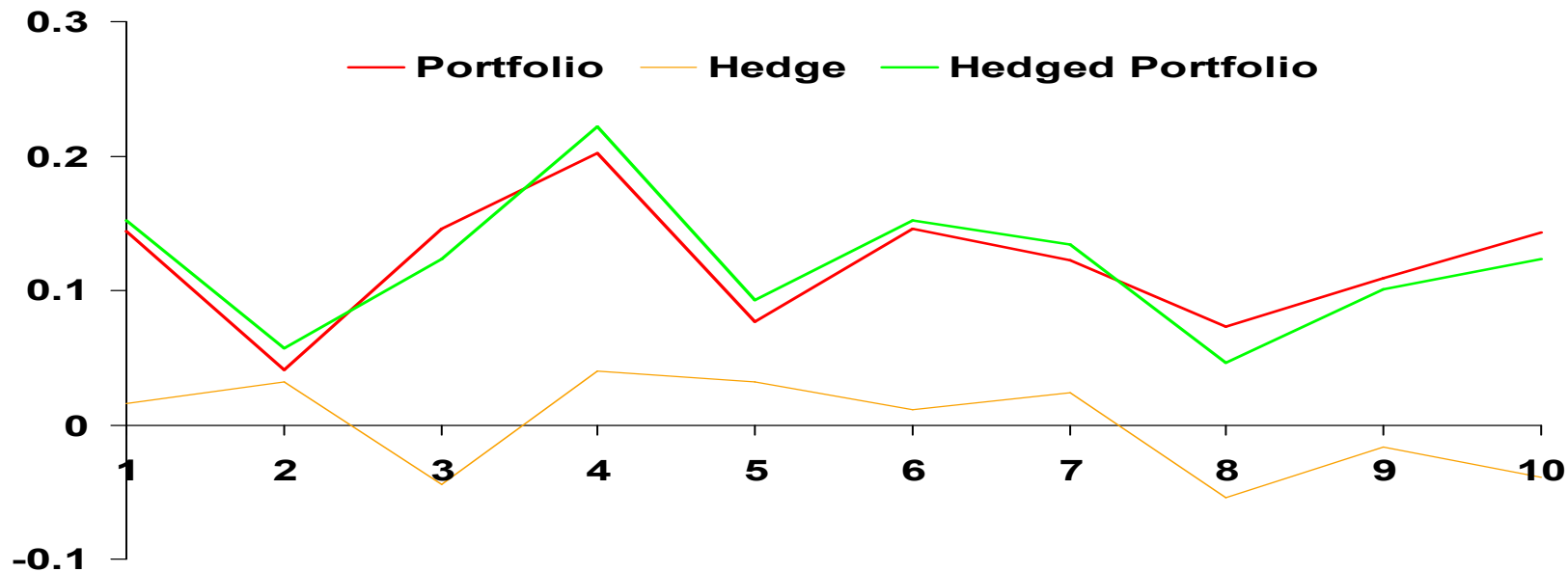


- Construct a portfolio from two assets that are negatively correlated in each period. Within period variance is less than the variance of each asset.
- The distribution of the return histories over n periods has mean and variance equal to n times the within period mean and variance.

Distribution of Return Histories

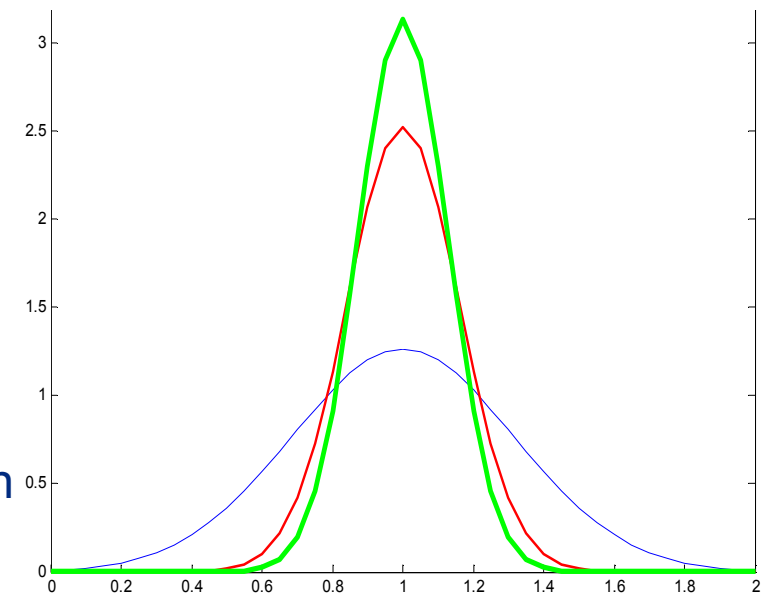


Time diversification of risk

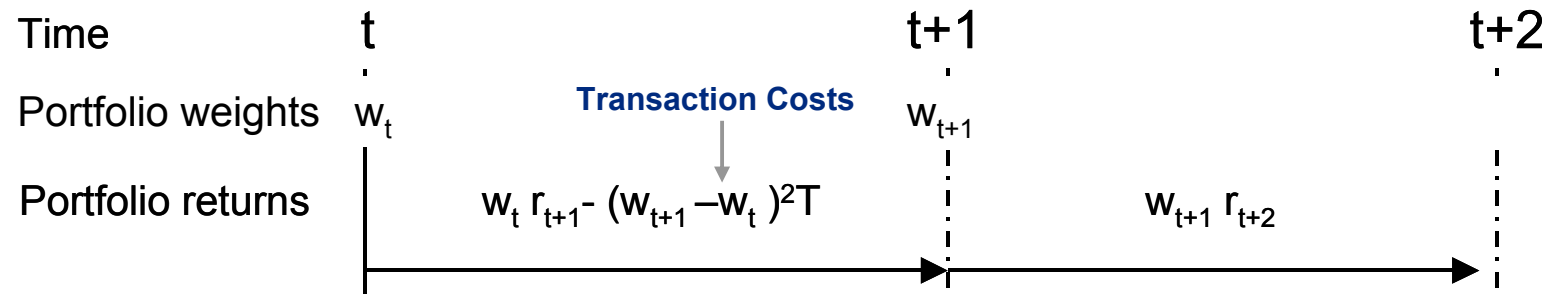


- A dynamic hedge is a portfolio whose returns are negatively correlated to original portfolio returns **across** periods.
- The within period variance of the aggregate portfolio can be greater than the variance of unhedged portfolio.
- **But** the distribution of aggregate returns over n periods will have mean and variance less than n times the within period mean and variance.

Distribution of Return Histories



Return Predictability and Transaction Costs can induce correlation across periods



- Let $r_{t+i}^e = E_{t+i-1}(r_{t+i})$ denote expected returns at the beginning of the period. Let the portfolio construction rule be $w_t = Kr_{t+1}^e$ and define the system equations as follows

$$r_{t+2}^e = Ar_{t+1}^e + \eta_{t+1}$$

$$r_{t+i} = r_{t+i}^e + \varepsilon_{t+i} \quad \text{for } i = 1, 2$$

where η , ε are i.i.d. random variables.

- Note the following
 - Returns are just equal to expected returns plus an innovation.
 - Expected returns are time-varying if $\eta \neq 0$ and predictable if $A \neq 0$.
 - The portfolio rule can be any function of expected returns – this is just the simplest.
 - Transaction cost can be any function of the change in portfolio weights.

Return Predictability and Transaction Costs can induce correlation across periods

- Recall the system equations

$$r_{t+2}^e = Ar_{t+1}^e + \eta_{t+1}$$

$$r_{t+1} = r_{t+1}^e + \varepsilon_{t+1}$$

$$r_{t+2} = r_{t+2}^e + \varepsilon_{t+2} = Ar_{t+1}^e + \eta_{t+1} + \varepsilon_{t+2} \dots \dots$$

- The returns to the portfolio in periods 1 and 2 are:

$$\text{Return period 1} := w_t r_{t+1} - T(w_{t+1} - w_t)^2 = Kr_{t+1}^e (r_{t+1}^e + \varepsilon_{t+1}) - TK((A-1)r_{t+1}^e + \eta_{t+1})^2$$

$$\text{Return period 2} := w_{t+1} r_{t+2} = K(Ar_{t+1}^e + \eta_{t+1})(Ar_{t+1}^e + \eta_{t+1} + \varepsilon_{t+2})$$

- And so returns are correlated across periods if:

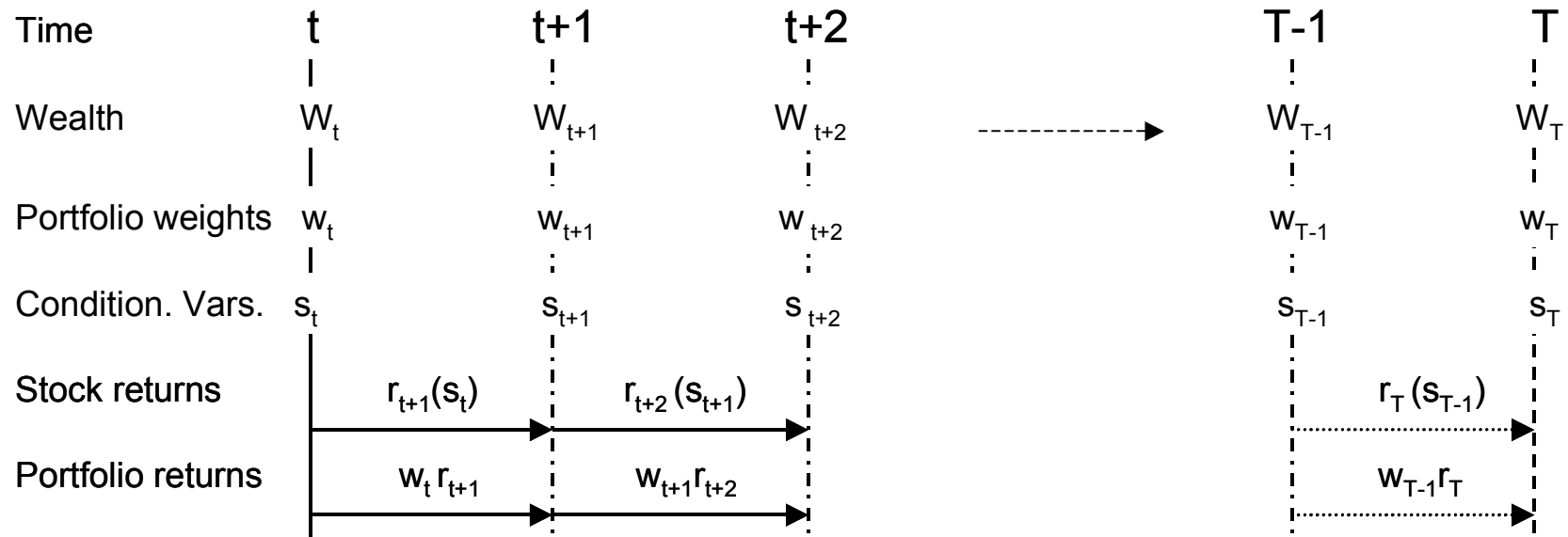
Case 1: $E(\varepsilon_{t+1}\eta_{t+1}) \neq 0$ Merton hedging demands

Case 2: $T \neq 0$ and $\eta_{t+1} \neq 0$ Varying returns and transaction costs

The Hard Sell

- The approach gives an explicit solution to the dynamic portfolio construction problem. Thus it can be used to investigate:
 1. How to combine forecasts with different decay rates, for example
 - in a diversified equity portfolio - A value factor forecast with a 9 half-life, sector momentum factors with a 3 month half-life, idiosyncratic short term reversals with 2 week half-life.
 - In asset allocation problem – a short-term price momentum, with a longer term price reversal.
 - A bond portfolio with a structural/consistent model of the yield curve.
 2. Smoothing long-term performance using dynamic forecast hedging, for example
 - Pension portfolio that takes in account of any long-term mean-reversion in asset returns

The Dynamic Portfolio Construction Problem



- The **Forecasting Equation** describes the time evolution of the conditioning variables (or states)

$$s_{t+1} = A_s s_t + \eta_{t+1} + \mu_s$$

- The **Output Equation** describes the relationship between the states and gross log returns

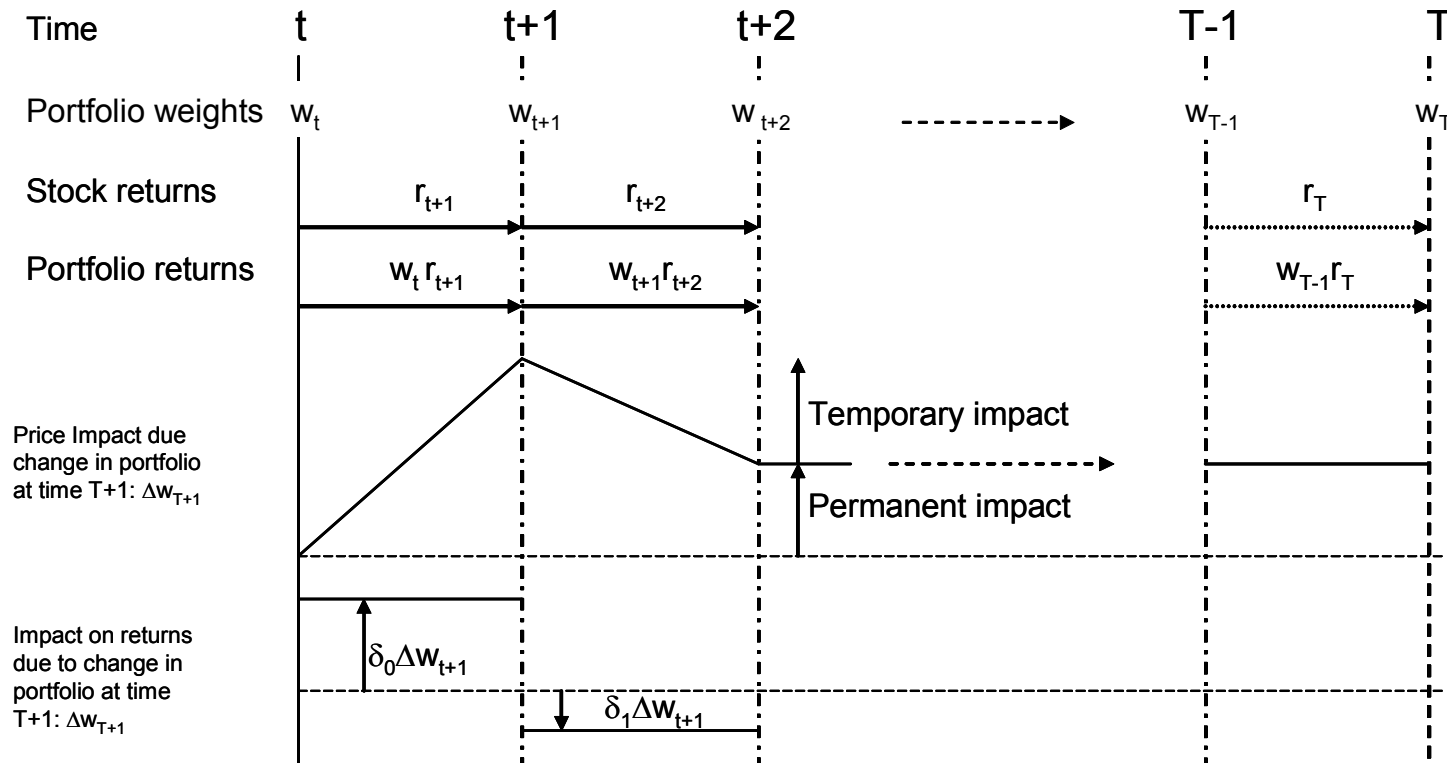
$$r_{t+1}^{Gross} = C_s s_t + \varepsilon_{t+1} + \kappa_s$$

Modelling Transaction Costs and Net Returns

- We model both the explicit cost (Grinold 2006) as

$$C_t^{Explicit} = -\Delta w_t' T \Delta w_t$$

- And the implicit cost (Almgren & Chriss 2000, Engle & Ferstenberg, 2006) due to the price impact of the trade



$$r_{t+1} = r_{t+1}^{Gross} + \delta_0 \Delta w_{t+1} - \delta_1 \Delta w_t$$

Putting it together

- The **Forecasting Equation** is augmented by the change in portfolio positions

$$\begin{bmatrix} s_{t+1} \\ w_{t+1} \\ \Delta w_{t+1} \end{bmatrix} = \begin{bmatrix} A_s & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_t \\ w_t \\ \Delta w_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \eta_{t+1} + \begin{bmatrix} 0 \\ I \\ I \end{bmatrix} \Delta w_{t+1} + \begin{bmatrix} \mu_s \\ 0 \\ 0 \end{bmatrix}$$

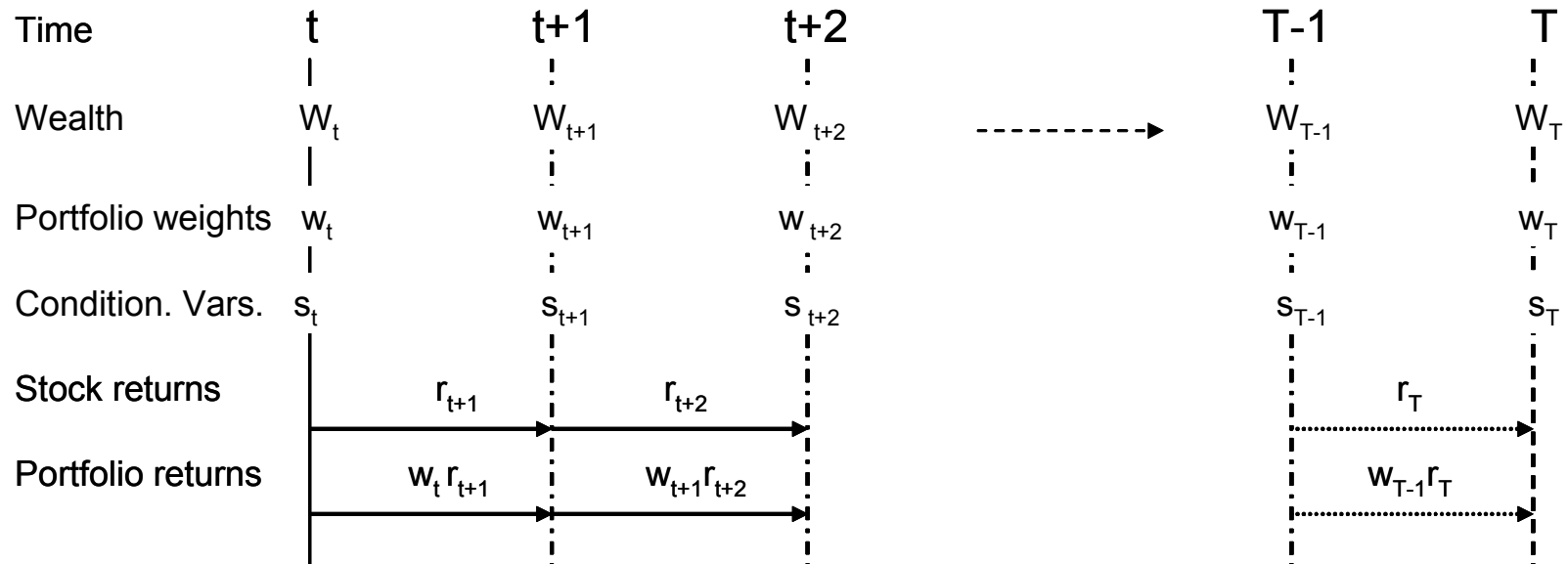
We shall refer to the vector $x_t = \begin{bmatrix} s_t^T & w_t^T & \Delta w_t^T \end{bmatrix}^T$ as the states.

- So that the **Output equation** includes the implicit trading costs

$$z_t = \begin{bmatrix} r_{t+1} \\ w_t \\ \Delta w_{t+1} \end{bmatrix} = \begin{bmatrix} C_s & 0 & -\delta_1 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_t \\ w_t \\ \Delta w_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \varepsilon_t + \begin{bmatrix} \delta_0 \\ 0 \\ I \end{bmatrix} \Delta w_{t+1} + \begin{bmatrix} \kappa_s \\ 0 \\ 0 \end{bmatrix}$$

- Explicit Linear Portfolio Constraints (e.g. fully-invested) can be included too

The Mean-Variance Performance Criterion



- The problem is to choose portfolios, w_t , to maximise the performance measure

$$\frac{1}{1-\gamma} \log \left[E_t \left(W_T^{1-\gamma} \right) \right]$$

- This is just mean-variance extended to the multi-period problem. For if

$$W_T = W_t e^r \approx W_t (1+r) \quad \text{then}$$

$$\frac{1}{1-\gamma} \log E_t \left(W_T^{1-\gamma} \right) = \log W_t + E(r) + \frac{1-\gamma}{2} \text{Var}(r)$$

A Tractable Optimisation Criterion

- We use Campbell & Viceria (2003) log-linear approximation. Let $\text{Var}(r_t)=\Sigma$ and $\sigma^2=\text{diag}(\Sigma)$

$$W_T = W_t \left(w_t' e^{r_{t+1}} \right) \left(w_{t+1}' e^{r_{t+2}} \right) \dots$$

$$\approx W_t \left(e^{w_t' r_{t+1} + \frac{1}{2} w_t' (\sigma^2 - \Sigma w_t)} \right) \left(e^{w_{t+1}' r_{t+2} + \frac{1}{2} w_{t+1}' (\sigma^2 - \Sigma w_{t+1})} \right) \dots = W_t \left(e^{\sum_{i=0}^{T-1} w_{t+i}' r_{t+1+i} + \frac{1}{2} w_{t+i}' (\sigma^2 - \Sigma w_{t+i})} \right)$$

- And substitute this into our performance measure

$$\frac{1}{1-\gamma} \log \left[E_t \left(W_T^{1-\gamma} \right) \right] = \log W_t + \frac{1}{1-\gamma} \log \left[E_t \left(e^{\sum_{i=0}^{T-1} w_{t+i}' r_{t+1+i} + \frac{1}{2} w_{t+i}' (\sigma^2 - \Sigma w_{t+i})} \right) \right]$$

- This is an exponential of a quadratic cost or risk-sensitive control problem studied in

Peter Whittle (1990). *Risk Sensitive Optimal Control*, Wiley & Sons.

Iglesias, P.A. and Glover, K. (1991). 'State-space approach to discrete-time H_∞ Control', *International Journal of Control*, Vol 54(5), pp1031-1073.

The Performance Criterion

- The cost function is a function of these performance variables

$$\begin{aligned}
 \mathbb{C}_t^{Total} &= \underbrace{w_t^T r_{t+1} + \frac{1}{2} w_t^T (\sigma^2 - \Sigma w_t)}_{\text{Net Log Return to Portfolio}} - \underbrace{\Delta w_t^T T \Delta w_t}_{\text{Explicit Costs}} \\
 &= \begin{bmatrix} r_{t+1} \\ w_t \\ \Delta w_{t+1} \end{bmatrix}^T \begin{bmatrix} 0 & \frac{1}{2} I & 0 \\ \frac{1}{2} I & -\frac{1}{2} \Sigma & 0 \\ 0 & 0 & -T \end{bmatrix} \begin{bmatrix} r_{t+1} \\ w_t \\ \Delta w_{t+1} \end{bmatrix} - 2 \begin{bmatrix} 0 \\ -\frac{1}{4} \sigma^2 \\ 0 \end{bmatrix}^T \begin{bmatrix} r_{t+1} \\ w_t \\ \Delta w_{t+1} \end{bmatrix} \\
 &= z_t^T R z_t - 2 S^T z_t
 \end{aligned}$$

- The problem is to choose Δw_t so as to maximise the criterion

$$U_t = \frac{2}{1-\gamma} \log_e E_t \left(e^{(1-\gamma) \sum_{i=t+1}^T \mathbb{C}_{i-1}} \right)$$

Why is it tractable?

- The tractability follows directly from *Lemma 6.1.2*, Whittle (1990). If $\mathbb{C}(w,r,\varepsilon)$ is a quadratic function in all 3 arguments and positive definite in ε then

$$\min_w \log E_{\varepsilon} \left(e^{-\frac{1}{2}\mathbb{C}(r,w,\varepsilon)} \right) \propto -\frac{1}{2} \max_w \min_{\varepsilon} \mathbb{C}(r, w, \varepsilon)$$

Hence the problem reduces to finding the best portfolio w in the face of the worst case disturbance ε .

- Others have studied similar problems in the finance literature but have not used the explicit solutions developed in the control literature.

Campbell, John and Viceria (2003). 'A multivariate model of asset allocation', Review of Financial Studies.

Bielecki, Stanley R. and Pliska (2005). 'Risk sensitive asset management with transaction costs', Finance and Stochastics

Litterman, Robert (2006). 'Multi –Period Portfolio Optimisation. Presentation to Risk Conference.

Solution to Dynamic Portfolio Construction Problem

- This is a minor extension of the results in Whittle (1990) and Iglesias and Glover (1991). For brevity rewrite out **forecasting** and **output** equations in terms of the n state vector x

$$\begin{aligned} x_{t+1} &= Ax_t + B_1\xi_{t+1} + B_2\Delta w_t + \mu \\ z_t &= Cx_t + D_1\xi_{t+1} + D_2\Delta w_t + \kappa \end{aligned} \quad \text{where} \quad \xi_{t+1} = \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix}$$

and $\text{Var}(\xi) = \Omega$. At any time t the achieved performance (the value function) can be expressed in terms of the $n \times n$ matrix Π and n -vector Φ and the states, x_t

$$\begin{aligned} \frac{1}{1-\gamma} \log \left[E_t \left(W_T^{1-\gamma} \right) \right] &= x_t^T \Pi_t x_t - 2\Phi_t^T x_t + \phi_t \\ &= \begin{bmatrix} s_t \\ w_t \\ \Delta w_t \end{bmatrix}^T \begin{bmatrix} \Pi_{11,t} & \Pi_{12,t} & \Pi_{13,t} \\ \Pi_{12,t}^T & \Pi_{22,t} & \Pi_{23,t} \\ \Pi_{13,t}^T & \Pi_{23,t}^T & \Pi_{33,t} \end{bmatrix} \begin{bmatrix} s_t \\ w_t \\ \Delta w_t \end{bmatrix} - 2 \begin{bmatrix} \Phi_{1,t} \\ \Phi_{2,t} \\ \Phi_{3,t} \end{bmatrix}^T \begin{bmatrix} s_t \\ w_t \\ \Delta w_t \end{bmatrix} + \phi_t \end{aligned}$$

The matrix Π , and Φ are solved by backward induction.

Solution to Dynamic Portfolio Construction Problem (2)

Set $\Pi_T=0$, and $\Phi_T=0$. Now define

$$V_{t+1} = \begin{bmatrix} D_1^T \\ D_2^T \end{bmatrix} R \begin{bmatrix} D_1 & D_1 \end{bmatrix} + \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \Pi_{t+1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} ((\gamma-1)\Omega)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$K_{t+1} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = -V^{-1} \left(\begin{bmatrix} D_1^T \\ D_2^T \end{bmatrix} RC + \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \Pi_{t+1} A \right)$$

$$W_{t+1} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = V^{-1} \left(\begin{bmatrix} D_1^T \\ D_2^T \end{bmatrix} S + \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} (\Phi_{t+1} - \Pi_{t+1} \mu) \right)$$

then matrix Π is a solution to the following H_∞ Riccati equation

$$\Pi_t = A^T \Pi_{t+1} A + C^T RC - K_{t+1}^T V_{t+1}^{-1} K_{t+1}$$

and Φ solves the linear equation

$$\Phi_t = C^T S + A^T (\Phi_{t+1} - \Pi_{t+1} \mu) + K_{t+1}^T V_{t+1} W_{t+1}$$

Solution to Dynamic Portfolio Construction Problem (3)

- These equations are solved by backward iteration.
 - The steady state can also be found by solving an eigenvector-eigenvalue problem or by iterating until convergence

- The optimal portfolio weights is then calculated as

$$w_t = -\left(\Pi_{22,t} - (\delta_1 + \delta_1^T + 2T)\right)^{-1} \left(\Pi_{12,t}^T s_t - \Phi_{2,t} + (\delta_1 + \delta_0^T + 2T)w_{t-1}\right)$$

This has a very familiar form. An adjusted covariance matrix times a vector of risk adjusted expected returns

Strategic Asset Allocation – Barberis (2000)

- An allocator chooses between investing in a risk free asset with return r_f and risky equity asset with excess returns, r_t .
- The dividend price ratio (d/p) forecasts future equity returns. The following system describes equity returns

$$\begin{bmatrix} (d/p)_{t+1} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} a_s \\ c_s \end{bmatrix} (d/p)_t + \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix} + \begin{bmatrix} \mu_s \\ \kappa_s \end{bmatrix}$$

$$\begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\eta^2 & \rho\sigma_\mu\sigma_\varepsilon \\ \rho\sigma_\mu\sigma_\varepsilon & \sigma_\varepsilon^2 \end{bmatrix} \right)$$

where $a_s < 1$ and $\rho < 0$.

On monthly US data between 1986-1995, $a_s = 0.96$, $\rho = -0.93$ (and $c_s = 1.0919$, $\mu_s = 0.0013$, $\kappa_s = -0.0303$, $\sigma_\eta^2 = 2.6E-6$, $\sigma_\varepsilon^2 = 0.0019$, $r_f = 0.0036$)

- Note ε_t is negatively correlated with such a shock \Rightarrow Equity is a hedge against a fall in future returns.
- The problem is to choose the portfolio allocation to equity, w_t

Solution to the asset allocation problem

- The value function is quadratic in the dividend ratio $(d/p)_t$

$$\frac{1}{1-\gamma} \log \left[E_t \left(W_T^{1-\gamma} \right) \right] = \Pi_t \left(\frac{d}{p} \right)_t^2 - 2\Phi_t \left(\frac{d}{p} \right)_t + \phi_t$$

Both Π_t and Φ_t are found by backward iteration, with initial values $\Pi_T = \Phi_T = 0$, where

Define $\sigma_{\eta|\varepsilon,t+1}^2 = \frac{1}{2} \Pi_{t+1} \left(1 + 2(\gamma-1) \Pi_{t+1} \sigma_\eta^2 \left(1 - \frac{(\gamma-1)}{\gamma} \rho^2 \right) \right)$ then

$$\Pi_t = \frac{1}{2} \frac{c_s^2}{\gamma \sigma_\varepsilon^2} + \frac{1}{2} \frac{\Pi_{t+1}^2}{\sigma_{\eta|\varepsilon,t+1}^2} \left(a_s - \frac{(\gamma-1)}{\gamma} c_s \rho \frac{\sigma_\eta}{\sigma_\varepsilon} \right)^2$$

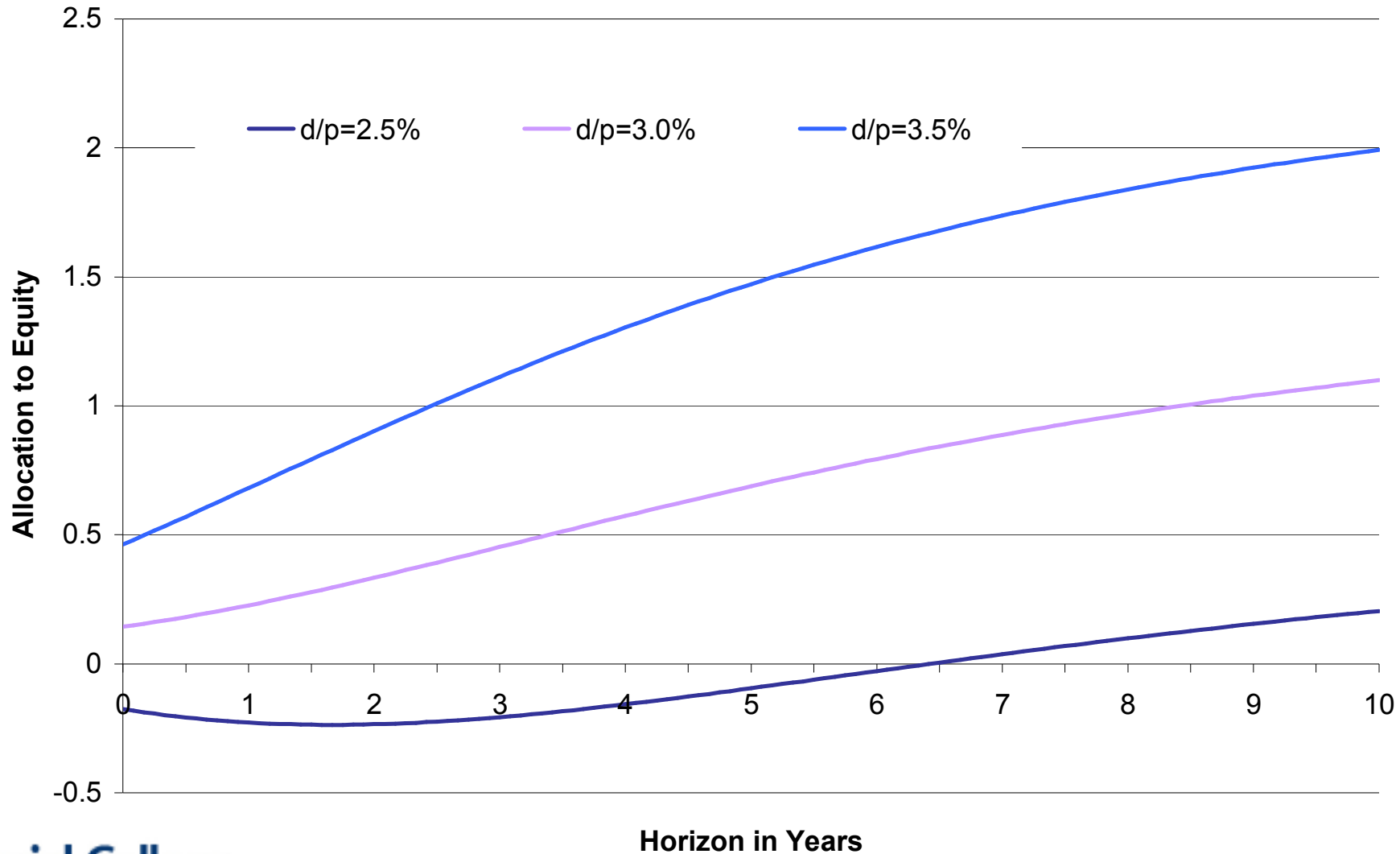
$$\Phi_t = -\frac{1}{2} \frac{c_s}{\gamma \sigma_\varepsilon^2} \left(\kappa_s + \frac{\sigma_\varepsilon^2}{2} \right) - \frac{1}{2} \frac{\Pi_{t+1}^2}{\sigma_{\eta|\varepsilon,t+1}^2} \left(a_s - \frac{(\gamma-1)}{\gamma} c_s \rho \frac{\sigma_\eta}{\sigma_\varepsilon} \right) \left(\left(\Pi_{t+1} \mu_s - \Phi_{t+1} \right) - \frac{(\gamma-1)}{\gamma} \Pi_{t+1} \rho \frac{\sigma_\eta}{\sigma_\varepsilon} \left(\kappa_s + \frac{\sigma_\varepsilon^2}{2} \right) \right)$$

- The optimal allocation to equity is

$$w_t = \underbrace{\frac{\left(c_s s_t + \kappa_s + \frac{\sigma_\varepsilon^2}{2} \right)}{\gamma \sigma_\varepsilon^2}}_{\text{Mean-Variance}} - \underbrace{\frac{(\gamma-1)}{\gamma} \rho \frac{\sigma_\eta}{\sigma_\varepsilon} \left(\frac{\Pi_{t+1}}{\sigma_{\eta|\varepsilon,t+1}^2} \right) \left(\Pi_{t+1} \left(a_s s_t + \mu_s \right) - \frac{(\gamma-1)}{\gamma} \rho \frac{\sigma_\eta}{\sigma_\varepsilon} \left(c_s s_t + \kappa_s + \frac{\sigma_\varepsilon^2}{2} \right) \right) - \Phi_{t+1}}_{\text{Hedging Portfolio}}$$

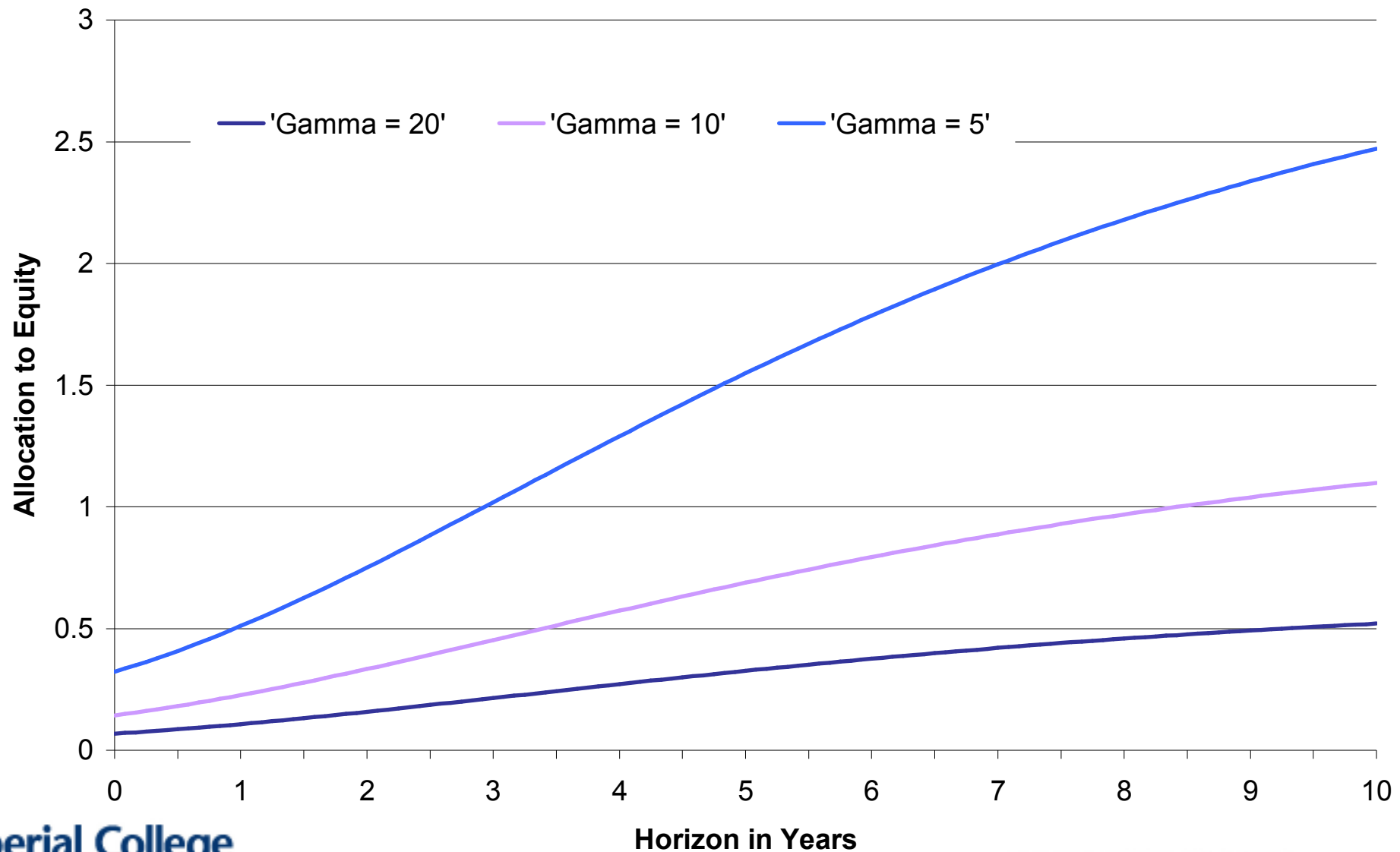
The longer the horizon, the greater allocation to equity

- The allocation to equity as a function of both the dividend / price ratio and the investment horizon given $\gamma = 10$.



The more risk averse, the lower the allocation to equity

- The allocation to equity as a function of the coefficient of risk aversion and the investment horizon for a dividend/price ratio of 3%



Performance Comparison

- Comparison of three portfolio construction techniques over a 10 year horizon
 - We compare the average annualised log of the excess returns over the 10 years
 - The risk aversion parameter of single period mean variance rule is chosen so as to have the same sample variance as the dynamic optimisation portfolio.

	100% Equity Allocation	Single Period Mean Variance	Dynamic Optimisation
		$\gamma = 5$	
Mean Excess Return	0.024	0.128	0.146
Std. Dev.	0.073	0.155	0.155
Sharpe	0.330	0.826	0.947
		$\gamma = 10$	
Mean Excess Return	0.024	0.058	0.066
Std. Dev.	0.070	0.067	0.067
Sharpe	0.347	0.856	0.977
		$\gamma = 20$	
Mean Excess Return	0.024	0.028	0.032
Std. Dev.	0.071	0.033	0.033
Sharpe	0.338	0.861	0.950

The Forecast Horizon Problem

- Assume there are two assets, v (a value portfolio) and m (a momentum portfolio). The **Forecasting equations** is

$$\begin{bmatrix} m_{t+1} \\ w_{t+1}^m \\ v_{t+1} \\ w_{t+1}^v \end{bmatrix} = \begin{bmatrix} 1 - \lambda_m & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - \lambda_v & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_t \\ w_t^m \\ v_t \\ w_t^v \end{bmatrix} + \begin{bmatrix} \lambda_m \eta_{t+1}^m \\ 0 \\ \lambda_v \eta_{t+1}^v \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta w_{t+1}^m \\ \Delta w_{t+1}^v \end{bmatrix}$$

- We shall assume that all but the permanent price impact are zero. Hence the **Output equation** is

$$\begin{bmatrix} r_{t+1}^m \\ w_t^m \\ r_{t+1}^v \\ w_t^v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_t \\ w_t^m \\ v_t \\ w_t^v \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1}^m \\ 0 \\ \varepsilon_{t+1}^v \\ 0 \end{bmatrix} + \begin{bmatrix} \delta_0^m & 0 \\ 0 & 0 \\ 0 & \delta_0^v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta w_{t+1}^m \\ \Delta w_{t+1}^v \end{bmatrix}$$

- To simplify the problem assume all innovations are independent
- The problem is to choose portfolio holdings, w_t^v and w_t^m in the two assets.

Solution to the forecast horizon problem

- The value function is quadratic in the states

$$\frac{1}{1-\gamma} \log \left[E_t \left(W_T^{1-\gamma} \right) \right] = \frac{\begin{bmatrix} m_t \\ w_t^m \\ v_t \\ w_t^v \end{bmatrix}^T \begin{bmatrix} \Pi_{m11,t} & \Pi_{m12,t} & 0 & 0 \\ \Pi_{m12,t} & \Pi_{m22,t} & 0 & 0 \\ \hline 0 & 0 & \Pi_{v11,t} & \Pi_{v12,t} \\ 0 & 0 & \Pi_{v12,t} & \Pi_{v22,t} \end{bmatrix} \begin{bmatrix} m_t \\ w_t^m \\ v_t \\ w_t^v \end{bmatrix} + \frac{\begin{bmatrix} \Phi_{m1,t} \\ \Phi_{m2,t} \\ \Phi_{v1,t} \\ \Phi_{v2,t} \end{bmatrix}^T \begin{bmatrix} m_t \\ w_t^m \\ v_t \\ w_t^v \end{bmatrix}}{\begin{bmatrix} \Phi_{m1,t} \\ \Phi_{m2,t} \\ \Phi_{v1,t} \\ \Phi_{v2,t} \end{bmatrix}^T \begin{bmatrix} m_t \\ w_t^m \\ v_t \\ w_t^v \end{bmatrix}} + \phi_t$$

Both Π_t and Φ_t are found by backward iteration, with initial values $\Pi_T = \Phi_T = 0$. As the innovations are independent, Π_t has a block diagonal structure. We just look at the top right block. Let

$$\sigma_{m,\eta|\varepsilon,t+1}^2 = \frac{\lambda_m^2 \sigma_m^2 \Pi_{m22,t+1}}{\frac{1}{2} \Pi_{m22,t+1} + (\gamma-1) \lambda_m^2 \sigma_m^2 |\Pi_{m,t+1}|} \quad \text{then}$$

$$\begin{bmatrix} \Pi_{m11,t} & \Pi_{m12,t} \\ \Pi_{m12,t} & \Pi_{m22,t} \end{bmatrix} = \begin{bmatrix} (1-\lambda_m)^2 \frac{|\Pi_{m,t+1}|}{\Pi_{m22,t+1}} & \frac{1}{2} \frac{\delta_m^0 (1-\lambda_m) \Pi_{m12,t+1}}{\Pi_{m22,t+1}} \\ \frac{1}{2} \frac{\delta_m^0 (1-\lambda_m) \Pi_{m12,t+1}}{\Pi_{m22,t+1}} & \left(\frac{\sigma_{m,\varepsilon}^2}{2} + \delta_m^0 + \frac{(\delta_m^0)^2}{4 \Pi_{m22,t+1}} \right) \end{bmatrix} - (\gamma-1) \sigma_{m,\eta|\varepsilon,t+1}^2 \begin{bmatrix} (1-\lambda_m) \frac{|\Pi_{m,t+1}|}{\Pi_{m22,t+1}} \\ \frac{\delta_m^0 \Pi_{m12,t+1}}{2 \Pi_{m22,t+1}} \end{bmatrix} \begin{bmatrix} (1-\lambda_m) \frac{|\Pi_{m,t+1}|}{\Pi_{m22,t+1}} & \frac{\delta_m^0 \Pi_{m12,t+1}}{2 \Pi_{m22,t+1}} \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{m1,t} \\ \Phi_{m2,t} \end{bmatrix} = \begin{bmatrix} (1-\lambda_m) \left(\Phi_{m1,t} \frac{\Pi_{m12,t+1}}{\Pi_{m22,t+1}} \Phi_{m2,t} \right) \\ \frac{\sigma_{m,\varepsilon}^2}{4} \frac{\delta_m^0}{2 \Pi_{m22,t+1}} \Phi_{m2,t} \end{bmatrix} - (\gamma-1) \sigma_{m,\eta|\varepsilon,t+1}^2 \begin{bmatrix} (1-\lambda_m) \frac{|\Pi_{m,t+1}|}{\Pi_{m22,t+1}} \\ \frac{\delta_m^0 \Pi_{m12,t+1}}{2 \Pi_{m22,t+1}} \end{bmatrix} \begin{bmatrix} \Phi_{m1,t} \frac{\Pi_{m12,t+1}}{\Pi_{m22,t+1}} \Phi_{m2,t} \end{bmatrix}$$

Solution to the forecast horizon problem (2)

- The optimal portfolio can be written in terms of the solution to these equations. However it is more instructive to look at the steady state or long horizon solution.
- When $\gamma=1$ (risk neutral) then the optimal steady-state portfolio can be written as

$$w_t^m = \frac{\sigma_{m\varepsilon}^2}{\tilde{\sigma}_{m\varepsilon}^2} \left(\left(\frac{\tilde{\sigma}_{m\varepsilon}^2}{4\delta_m^0 \lambda_m + \tilde{\sigma}_{m\varepsilon}^2} \right) \underbrace{\left(\frac{m_t}{\sigma_{m\varepsilon}^2} + \frac{1}{2} \right)}_{\text{Mean Variance Portfolio}} + \left(1 - \frac{\tilde{\sigma}_{m\varepsilon}^2}{4\delta_m^0 \lambda_m + \tilde{\sigma}_{m\varepsilon}^2} \right) \underbrace{\left(\frac{1}{2} \right)}_{\text{Long Run Portfolio}} \right) + \left(1 - \frac{\sigma_{m\varepsilon}^2}{\tilde{\sigma}_{m\varepsilon}^2} \right) w_{t-1}^m$$

Mean Variance Portfolio

Long Run Portfolio

$$w_t^v = \frac{\sigma_{v\varepsilon}^2}{\tilde{\sigma}_{v\varepsilon}^2} \left(\left(\frac{\tilde{\sigma}_{v\varepsilon}^2}{4\delta_v^0 \lambda_v + \tilde{\sigma}_{v\varepsilon}^2} \right) \underbrace{\left(\frac{m_t}{\sigma_{v\varepsilon}^2} + \frac{1}{2} \right)}_{\text{Mean Variance Portfolio}} + \left(1 - \frac{\tilde{\sigma}_{v\varepsilon}^2}{4\delta_v^0 \lambda_v + \tilde{\sigma}_{v\varepsilon}^2} \right) \underbrace{\left(\frac{1}{2} \right)}_{\text{Long Run Portfolio}} \right) + \left(1 - \frac{\sigma_{v\varepsilon}^2}{\tilde{\sigma}_{v\varepsilon}^2} \right) w_{t-1}^v$$

Mean Variance Portfolio

Long Run Portfolio

where $\tilde{\sigma}_{m\varepsilon}^2 = \frac{\sigma_{m\varepsilon}^2}{2} \left(1 + \sqrt{1 + \frac{4\delta_m^0}{\sigma_{m\varepsilon}^2}} \right)$ and $\tilde{\sigma}_{v\varepsilon}^2 = \frac{\sigma_{v\varepsilon}^2}{2} \left(1 + \sqrt{1 + \frac{4\delta_v^0}{\sigma_{v\varepsilon}^2}} \right)$

- The desired portfolio is a weighted average of the short and long run portfolios where the weight depends on speed of mean reversion λ and costs. The optimal portfolio is then a weighted average of previous period portfolio and the desired where the weight depends on costs only.

An Complete Example

- Construct a portfolio of the following 3 assets subject to being fully invested
 - Long-Short US Momentum Portfolio
 - Long-Short US Value Portfolio.
 - US Market Portfolio (fully invested constraint implies a weight of 1 on the market).
- The forecast model of expected returns was constructed as follows:
 1. Estimate a VAR (vector autoregression) on the returns to the portfolios and the following conditioning variables: Short rates, Credit and Term Spreads, Industrial Production, Inflation, Dividend Yields, X-section and Market Volatility on monthly data 1984 – 2005.
 2. From this VAR construct a time-series of forecast of expected returns to the portfolios.
 3. From this time series of forecasts estimate another VAR describing the evolution of these forecasts.

The Forecasting Model

- The state-space evolution equations is

$$\begin{bmatrix} r_{t+2|t+1}^{Mom} \\ r_{t+2|t+1}^{Val} \\ r_{t+2|t+1}^{Mkt} \\ s_{t+1}^{Term} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix} = \begin{bmatrix} 0.45 & 0.00 & -0.04 & 0 & 0 & 0 & 0 & 0 \\ 0.00 & 0.21 & -0.24 & 2.91 & 0 & 0 & 0 & 0 \\ 0.2 & -0.1 & 0.68 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.98 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{t+1|t}^{Mom} \\ r_{t+1|t}^{Val} \\ r_{t+1|t}^{Mkt} \\ s_t^{Term} \\ w_{t-1}^{Mom} \\ w_{t-1}^{Val} \\ \Delta w_{t-1}^{Mom} \\ \Delta w_{t-1}^{Val} \end{bmatrix} + \begin{bmatrix} \eta_{t+1}^{Mom} \\ \eta_{t+1}^{Val} \\ \eta_{t+1}^{Mkt} \\ \eta_{t+1}^{Term} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix} + \begin{bmatrix} .0044 \\ .0022 \\ .0025 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Where

$$Var \begin{bmatrix} \eta_{t+1}^{Mom} \\ \eta_{t+1}^{Val} \\ \eta_{t+1}^{Mkt} \\ \eta_{t+1}^{Term} \end{bmatrix} = \begin{bmatrix} \mathbf{1.10\%} & -0.08 & 0.52 & -0.09 \\ - & \mathbf{2.73\%} & -0.5 & -0.024 \\ - & - & \mathbf{2.86\%} & 0.4 \\ - & - & - & \mathbf{.05\%} \end{bmatrix}$$

Note

- Low variance of expected momentum returns
- Value and the Market returns are negatively correlated

The Model Output

- The output equation are

$$\begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{t+1|t}^{Mom} \\ r_{t+1|t}^{Val} \\ r_{t+1|t}^{Mkt} \\ s_t^{Term} \\ w_{t-1}^{Mom} \\ w_{t-1}^{Val} \\ \Delta w_{t-1}^{Mom} \\ \Delta w_{t-1}^{Val} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1}^{Mom} \\ \varepsilon_{t+1}^{Val} \\ \varepsilon_{t+1}^{Mkt} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.005 & 0 \\ 0 & -0.005 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix}$$

Where

$$Var \begin{bmatrix} \varepsilon_{t+1}^{Mom} \\ \varepsilon_{t+1}^{Val} \\ \varepsilon_{t+1}^{Mkt} \end{bmatrix} = \begin{bmatrix} \mathbf{13.61\%} & 0.03 & 0.02 \\ - & \mathbf{8.59\%} & -0.48 \\ - & - & \mathbf{14.58\%} \end{bmatrix}$$

Note

- High variance of the market and momentum returns
- Realised value and the Market returns are also negatively correlated

Hedging Opportunities

- To reduce variance of end of period wealth, investors would like to hold assets that deliver high returns when future investment opportunities decline.
- Therefore hedging involves the correlations between the innovations ε and ξ

$$Cor \left(\begin{bmatrix} \eta_{t+1}^{Mom} \\ \eta_{t+1}^{Val} \\ \eta_{t+1}^{Mkt} \\ \eta_{t+1}^{Term} \end{bmatrix}, \begin{bmatrix} \varepsilon_{t+1}^{Mom} \\ \varepsilon_{t+1}^{Val} \\ \varepsilon_{t+1}^{Mkt} \end{bmatrix} \right) = \begin{bmatrix} -.05 & 0.01 & -0.01 \\ -.59 & -.12 & 0.45 \\ .05 & -.53 & 0.09 \\ .19 & .05 & 0.06 \end{bmatrix}$$

- So
 1. Value is a hedge against the poor future market returns – When η^{Mkt} is negative, ε^{Val} tends to be positive
 2. Momentum is a hedge against poor returns to value – When η^{Val} is negative, ε^{Mom} tends to be positive

The Performance Criteria

- The cost function is a function of these output variables

$$\mathbb{C}_t^{Total} = r_{t+1}' w_t - \Delta w_t' T \Delta w_t$$

$$= \begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix}' \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix}$$

$$= z_t' R z_t - 2 S z_t$$

- Transaction costs are such that a 10% change in the allocation to value costs 20bp of return and a 5% change costs 5bp of return.

An Illustration

- We shall calculate the optimal portfolio allocations for this example when $\gamma=10$
- We shall compare our optimal dynamic portfolio to one calculated from the solution to the static problem of finding the optimal portfolio w_t given an initial portfolio w_{t-1} that maximises the Lagrangian

$$L = \left(w_t' r_{t+1|t}^e - (w_t - w_{t-1})' T (w_t - w_{t-1}) \right) - (\gamma - 1) w_t' \hat{\Omega} w_t$$

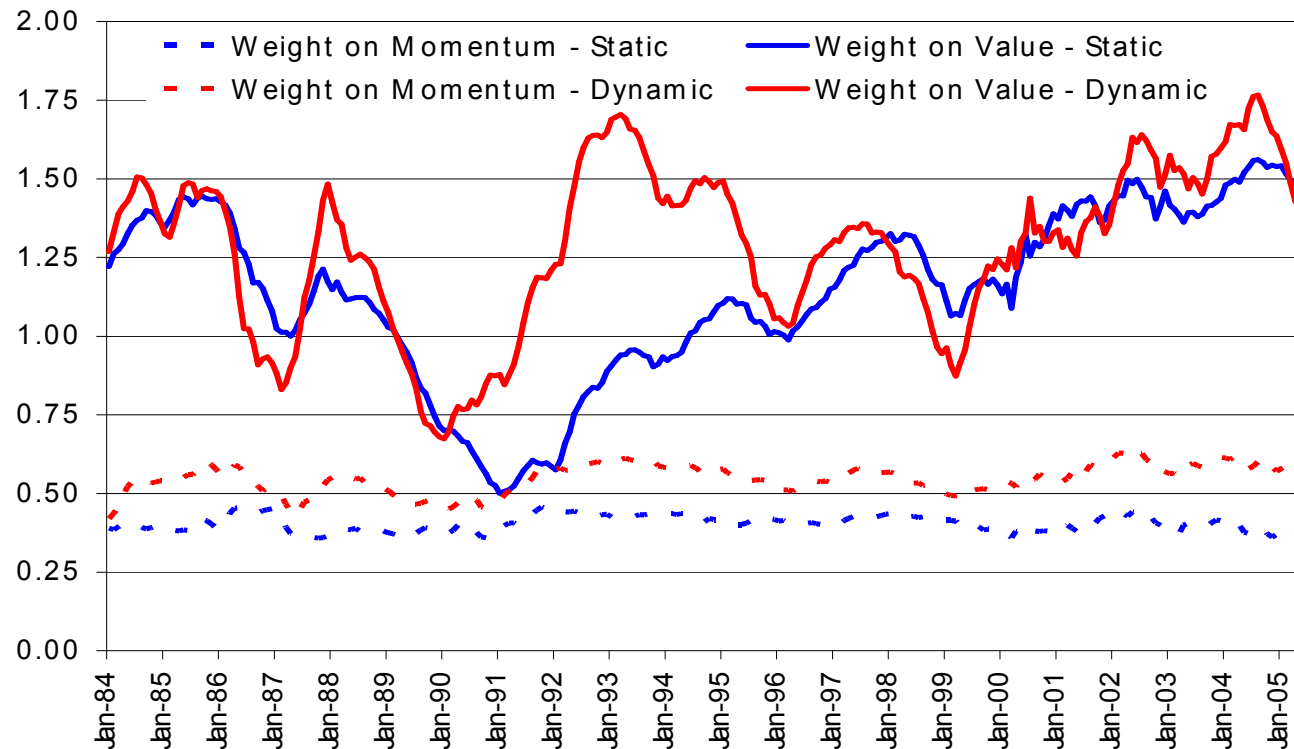
where

$$r_{t+1|t}^e = \begin{bmatrix} r_{t+1|t}^{Mom} \\ r_{t+1|t}^{Val} \\ r_{t+1|t}^{Mkt} \end{bmatrix} \quad \text{and} \quad \hat{\Omega} = Var(\varepsilon_t) \quad [= \Omega(5 : 7, 5 : 7)]$$

subject to the weight on the market being equal to 1.

A Comparison of the Static and Dynamic Approaches

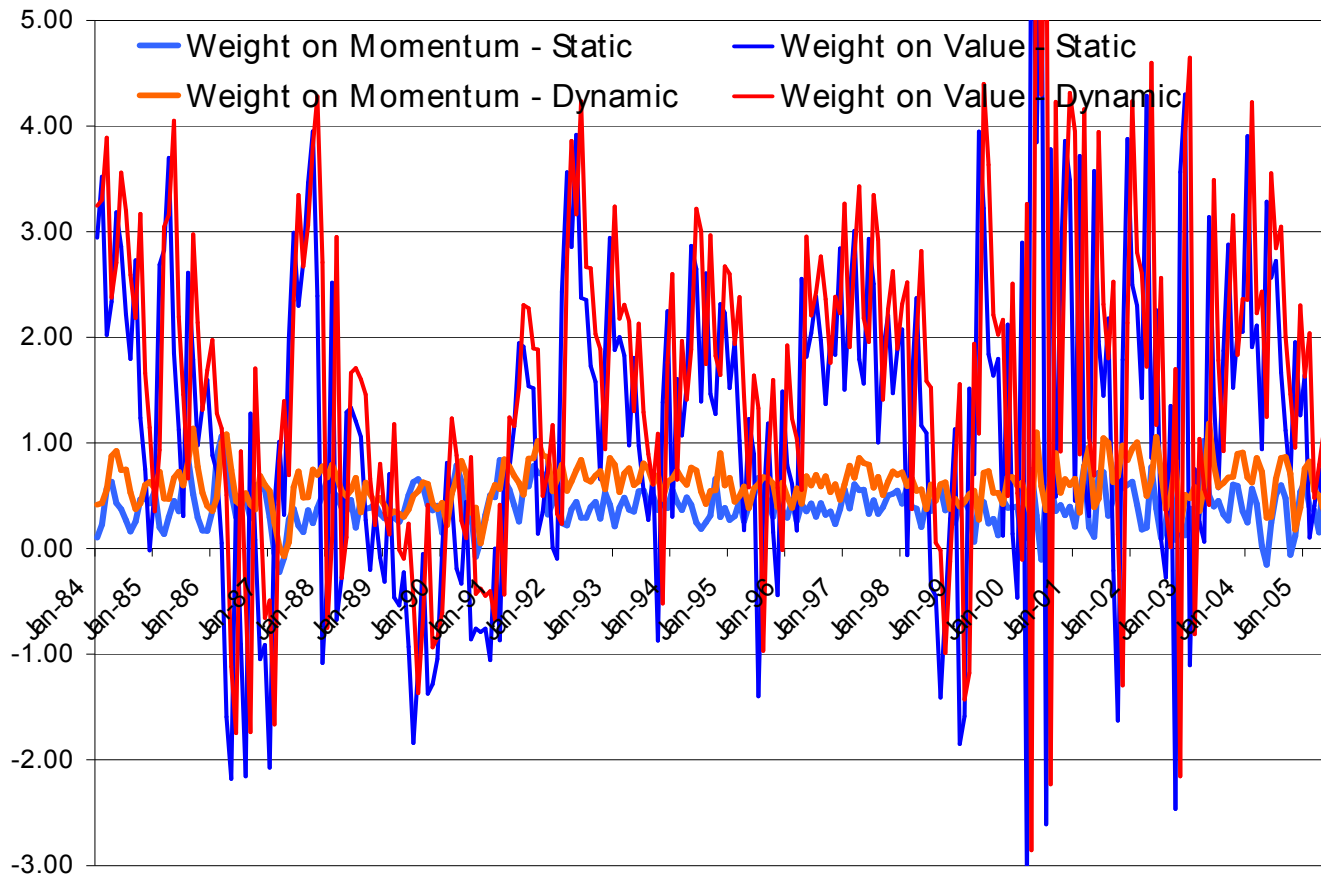
- We compare for the optimal portfolio weights over our data sample



1. The dynamic solution gives a far greater weight to Momentum
2. The dynamic solution plays value far harder during the value market 1992-1996

Deconstructing the solution - Hedging

- Set transactions costs to zero. Differences due to Hedging



1. Dynamic solution increases position on momentum as a hedge against value
2. Dynamic solution increases position on value as a hedge against the market.

Conclusions

- We have phrased the dynamic portfolio construction problem in the risk sensitivity framework of Whittle 1991. This framework is very flexible as allows us to model:
 - Forecast decay rates
 - Correlation in forecasts within and across periods
 - Transaction costs – both implicit and explicit
 - Classical Merton Portfolio Hedging motives
- We have shown that if
 - Preferences are described by constant relative risk aversion
 - The forecasting model is linear
 - The innovation are gaussianwe can calculate an explicit solution to the problem.
- We have illustrated the approach on a simple asset allocation problem, a simple forecast horizon problem and a more complex tactical equity strategy.