ROM Simulation: Exact Simulation using Random Orthogonal Matrices

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Outline

- Sampling Error in Monte Carlo Simulation
- Exact Covariance Simulation
- Multivariate Higher Moments and Orthogonal Matrices
- Theory of ROM Simulation
- ROM Simulation Techniques
- Example Applications: (a) Portfolio VaR and (b) Corridor Options

Sampling Error in Monte Carlo Simulation

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Example: Elliptical Monte Carlo

- Let the matrix $\mathbf{X}_{mn}$ represent a sample of size m on n stationary random variables. That is, the observations on the ith variable X_i, are in the ith column of $\mathbf{X}_{mn}$
- If the variables have an elliptical (multivariate normal or Student t) joint distribution, Monte Carlo simulation of $\mathbf{X}_{mn}$ has parameters $\mathbf{V}_n$, the covariance matrix and $\boldsymbol{\mu}_n$, the mean vector.
- That is, the ijth element of $\mathbf{V}_n$ is the covariance between the variables X_i and X_j, and the ith element of $\boldsymbol{\mu}_n$ is the mean of X_i
- Simulation: Given a *target* $\mathbf{V}_n$ and $\boldsymbol{\mu}_n$, we *want* to generate random samples $\mathbf{X}_{mn}$ such that

$$m^{-1}(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n')'(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n') = \mathbf{V}_n$$

- But, due to sampling error, the above relationship is only approximate

Simulation Accuracy Experiment

- Target is a symmetric circulant matrix of degree n with $\rho = 0.7$
- To measure the distance between a simulated sample correlation matrix, $\tilde{\mathbf{C}}_n$, and its corresponding target correlation matrix $\mathbf{C}_n$ we use the Frobenius norm:

$$||\tilde{\mathbf{C}}_n - \mathbf{C}_n||_{\mathbf{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |\tilde{c}_{ij} - c_{ij}|^2}$$

- (a) Multivariate normal, and Student t MC simulations with 6 degrees of freedom
- (b) ROM simulations (to be explained later in the talk)
- Accuracy measured as average of Frobenius norms over 1000 simulations

Results

- Frobenius norm values for ROM simulations have been multiplied by $\mathbf{10^{14}}$

	$n=5$		$n=10$		$n=50$	
Normal	ROM	MC	ROM	MC	ROM	MC
$m=10$	0.049	1.036				
$m=30$	0.100	0.560	0.232	1.046		
$m=90$	0.1822	0.323	0.5068	0.827	2.194	5.080
t_6	ROM	MC	ROM	MC	ROM	MC
$m=10$	0.050	1.110				
$m=30$	0.069	0.606	0.116	1.490		
$m=90$	0.139	0.351	0.319	0.855	1.017	5.010

Table: Average Frobenius norms for 1000 ROM and MC simulations

Sources of Sampling Error in Monte Carlo Simulation

In standard elliptical MC we:

- Find $\mathbf{A}_n$ satisfying $\mathbf{V}_n = \mathbf{A}_n'\mathbf{A}_n$,
- Set $\mathbf{X}_{mn} = \mathbf{Z}_{mn}\mathbf{A}_n + \mathbf{1}_m\boldsymbol{\mu}_n'$ where $\mathbf{Z}_{mn} \sim \mathrm{D}(\mathbf{0}_n, \mathbf{I}_n)$

Two sources of sampling error:

- In sample means, since $\bar{\mathbf{z}}_n \approx \mathbf{0}_n$
- This problem is easily overcome by setting

$$\mathbf{X}_{mn} = (\mathbf{Z}_{mn} - \mathbf{1}_m\bar{\mathbf{z}}_n')\mathbf{A}_n + \mathbf{1}_m\boldsymbol{\mu}_n'$$

- But also

$$m^{-1}(\mathbf{Z}_{mn} - \mathbf{1}_m\bar{\mathbf{z}}_n')'(\mathbf{Z}_{mn} - \mathbf{1}_m\bar{\mathbf{z}}_n') \approx \mathbf{I}_n$$

- So we only have the approximation that

$$m^{-1}(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n')'(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n') \approx \mathbf{A}_n'\mathbf{A}_n = \mathbf{V}_n$$

Exact Covariance Simulation

- Sampling Error in Monte Carlo Simulation
- **Exact Covariance Simulation**
- Multivariate Higher Moments and Orthogonal Matrices
- Theory of ROM Simulation
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Exact Covariance Simulation

- Set $\mathbf{Y}_{mn} = \mathbf{Z}_{mn} - \mathbf{1}_m \bar{\mathbf{z}}'_n$
- Transform $\mathbf{Y}_{mn}$ into a rectangular orthogonal matrix $\mathbf{W}_{mn}$
- That is, $\mathbf{W}'_{nm}\mathbf{W}_{mn} = \mathbf{I}_n$, so $\mathbf{W}_{mn}$ has orthogonal columns, each of unit length
- Set
$$\mathbf{X}_{mn} = m^{\frac{1}{2}}\mathbf{W}_{mn}\mathbf{A}_n + \mathbf{1}_m \boldsymbol{\mu}'_n$$
- Now the covariance matrix of $\mathbf{X}_{mn}$ equals $\mathbf{V}_n$ exactly
- So the problem of generating simulations which always have exactly the target covariance matrix reduces to finding an orthogonal transform of $\mathbf{Y}_{mn}$

Orthogonalization Methods

1. QR Factorization
   - Find $\mathbf{W}_{mn}$ by applying a standard technique, such as Gram-Schmidt orthogonalization, to $\mathbf{Y}_{mn}$
2. Spectral Decomposition
   - Set $\mathbf{Y}'_{mn}\mathbf{Y}_{mn} = \mathbf{Q}_n\boldsymbol{\Lambda}_n\mathbf{Q}_n^{-1}$ where $\boldsymbol{\Lambda}_n$ is the diagonal matrix of eigenvalues, and $\mathbf{Q}_n$ is the matrix of eigenvectors
   - Then set
   $$\mathbf{W}_{mn} = \mathbf{Y}_{mn}\mathbf{Q}_n\boldsymbol{\Lambda}_n^{-\frac{1}{2}}$$
   - Normalize the columns to have unit length

Example

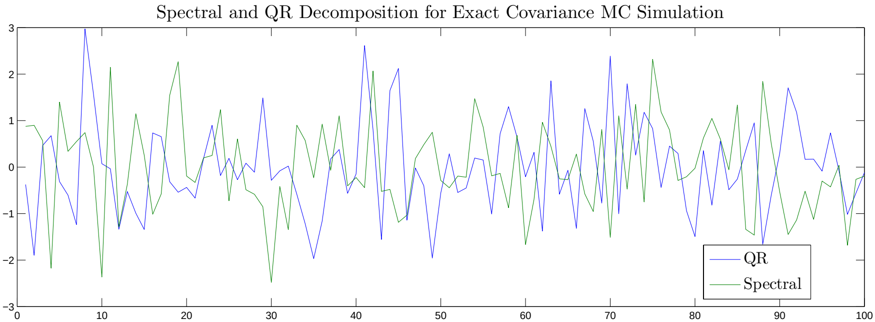

Figure: Both sample paths are generated from a standard MVN distribution with $n = 10$ and $m = 100$. Only the first random variable from each sample is shown, and in each case the multivariate sample has a mean of exactly zero and a covariance matrix exactly equal to the identity.

Motivation for ROM Simulation

Two main limitations of Monte Carlo Simulation:

- Orthogonalization only eliminates sampling error in means and covariance matrix, not in higher moments
- Monte Carlo is, essentially, a parametric method

Aims and Scope of ROM Simulation:

- Target multivariate skewness and kurtosis, in addition to exact means and covariance matrix
- Replicate dynamic features such as volatilty clustering, and features of marginal and terminal distributions, in a semiparametric framework

Multivariate Higher Moments and Orthogonal Matrices

Multivariate Skewness and Kurtosis

- We employ the multivariate measures introduced by Mardia (1970) which extend the standard univariate measures (e.g. Cramer, 1946):
- Kurtosis:

$$\kappa_M(\mathbf{X}_{mn}) = m^{-1}\sum_{i=1}^{m}\left\{(\mathbf{x}_i - \boldsymbol{\mu}_n')\mathbf{V}_n^{-1}(\mathbf{x}_i - \boldsymbol{\mu}_n')'\right\}^2$$

- Skewness:

$$\tau_M(\mathbf{X}_{mn}) = m^{-2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{(\mathbf{x}_i - \boldsymbol{\mu}_n')\mathbf{V}_n^{-1}(\mathbf{x}_j - \boldsymbol{\mu}_n')'\right\}^3$$

- Key Property is invariance under non-singular affine transformations:

$$\begin{aligned}\tau_M(\mathbf{X}_{mn}) &= \tau_M(\mathbf{X}_{mn}\mathbf{B}_n + \mathbf{1}_m\mathbf{b}_n)\\ \kappa_M(\mathbf{X}_{mn}) &= \kappa_M(\mathbf{X}_{mn}\mathbf{B}_n + \mathbf{1}_m\mathbf{b}_n)\end{aligned}$$

where $\mathbf{B}_n$ is any invertible matrix and $\mathbf{b}_n$ is any $1 \times n$ vector

Multivariate Skewness and Kurtosis: MVN

For the multivariate normal case, Mardia (1970) shows that

$$\mathbb{E}[\tau_M(\mathbf{X}_{mn})] = n(n+1)(n+2)m^{-1}$$

Therefore

$$\mathbb{E}[\tau_M(\mathbf{X}_{mn})] \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty$$

The corresponding results for kurtosis are

$$\mathbb{E}[\kappa_M(\mathbf{X}_{mn})] = n(n+2)(m-1)(m+1)^{-1}$$

Therefore

$$\mathbb{E}[\kappa_M(\mathbf{X}_{mn})] \longrightarrow n(n+2) \quad \text{as} \quad m \longrightarrow \infty$$

which is 3 when $n = 1$. We remark that

$$\mathbb{E}[\tau_M(\mathbf{X}_{mn})] = (n+1)(m+1)m^{-1}(m-1)^{-1}\mathbb{E}[\kappa_M(\mathbf{X}_{mn})]$$

Properties of Square Orthogonal Matrices

- A square matrix $\mathbf{Q}_n$ is orthogonal iff its inverse is equal to its transpose
- The covariance matrix of an orthogonal matrix is $m^{-1}\mathbf{I}_n$, since

$$\mathbf{Q}_n'\mathbf{Q}_n = \mathbf{Q}_n\mathbf{Q}_n' = \mathbf{I}_n$$

- Equivalently, both the rows and the columns of $\mathbf{Q}_n$ form an orthonormal set of vectors

Other properties include:

- $\det(\mathbf{Q}_n')\det(\mathbf{Q}_n) = 1 \quad \Longrightarrow \quad \det(\mathbf{Q}_n) = \pm 1$
- The product of two square orthogonal matrices is again square orthogonal

Properties of Rectangular Orthogonal Matrices

A rectangular matrix $\mathbf{Q}_{mn}$ is orthogonal iff

$$\mathbf{Q}'_{nm}\mathbf{Q}_{mn} = \mathbf{I}_n$$

It follows that $m \geq n$ and

- The column vectors, but not necessarily the row vectors, are orthonormal
- If $\mathbf{Q}_{mk}$ and $\mathbf{P}_{kn}$ are orthogonal then their product $\mathbf{W}_{mn} = \mathbf{Q}_{mk}\mathbf{P}_{kn}$ is also orthogonal
- There is a similar property for (normalized) sums of rectangular orthogonal matrices which are also ‘mututally’ orthogonal (Lemma 1.2 in paper)

Theory of ROM Simulation

- Sampling Error in Monte Carlo Simulation
- Exact Covariance Simulation
- Multivariate Higher Moments and Orthogonal Matrices
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Properties of Correlation Matrices

- If $\mathbf{C}_n = (c_{ij})$ is the correlation matrix of a sample $\mathbf{X}_{mn}$ then

$$\mathbf{C}_n = \mathbf{Y}'_{nm}\mathbf{Y}_{mn}$$

where $\mathbf{Y}_{mn} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$, $\mathbf{y}_i = m^{-\frac{1}{2}} s_i^{-1}(\mathbf{x}_i - \mathbf{1}_m \bar{x}_i)$, $\bar{x}_i$ and s_i are the mean and standard deviation of the ith variable, and $\mathbf{1}_m = (1, \ldots, 1)'$

- Hence, $\mathbf{C}_n$ is symmetric, positive semi-definite and $c_{ii} = \mathbf{y}_i{}'\mathbf{y}_i = 1$, so by virtue of Cauchy's inequality $|c_{ij}| = |\mathbf{y}_i{}'\mathbf{y}_j| \leq 1$
- A matrix with these properties is termed a *valid* correlation matrix
- But an additional *realisation condition* is also a necessary property of a sample correlation matrix:

$$\mathbf{1}_m{}'\mathbf{Y}_{mn} = \mathbf{0}_n$$

Definition of $\mathbf{L}_{mn}$

When $n = m - 1$, $\mathbf{L}_{mn}$ is:

$$\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{j(j+1)}} & \cdots & \cdots & \frac{1}{\sqrt{(m-1)m}} \\
\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & & \vdots & & & \vdots \\
0 & \frac{-2}{\sqrt{6}} & & \vdots & & & \vdots \\
\vdots & 0 & \ddots & \vdots & & & \vdots \\
\vdots & \vdots & \ddots & \frac{-j}{\sqrt{j(j+1)}} & & & \vdots \\
\vdots & \vdots & & 0 & \ddots & & \vdots \\
\vdots & \vdots & & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & \frac{-(m-1)}{\sqrt{(m-1)m}}
\end{pmatrix}$$

The matrix $\mathbf{L}_{mn}$ is obtained by deleting the first $m - 1 - n$ columns above

Structure of L-Matrices

The $\mathbf{L}_{mn}$ matrix is the Gram-Schmidt orthogonalization of the matrix whose jth column is the $m \times 1$ vector $(0, \ldots, 0, 1, -1, 0, \ldots, 0)'$

For example with $n = 3$ and $m = 4$:

$$\mathbf{L}_{4,3} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ 0 & 0 & -\frac{3}{\sqrt{12}} \end{pmatrix}$$

is the Gram-Schmidt orthogonalization of

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

L-Matrix Theorem (Walter Ledermann)

- Let $\mathbf{C}_n$ be a valid correlation matrix and set $\mathbf{C}_n = \mathbf{A}_n'\mathbf{A}_n$

Possibilities for $\mathbf{A}_n$:

- Cholesky decomposition: $\mathbf{A}_n$ is upper triangular
- Spectral decomposition: $\mathbf{A}_n = \mathbf{\Lambda}_n^{1/2}\mathbf{Q}_n$ where $\mathbf{Q}_n$ = matrix of eigenvectors and $\mathbf{\Lambda}_n$ = diagonal matrix of eigenvalues of $\mathbf{C}_n$

THEOREM:

- Let $\mathbf{R}_n$ be a random orthogonal matrix of degree n
- Set $\mathbf{W}_{mn} = \mathbf{L}_{mn}\mathbf{R}_n$ where $\mathbf{L}_{mn}$ is the $(m \times n)$ *L-matrix*
- Then $\mathbf{Y}_{mn} = \mathbf{W}_{mn}\mathbf{A}_n$ realises $\mathbf{C}_n$, and $\mathbf{Y}_{mn}$ satisfies the realisation condition (19)

ROM Simulation for Exact Covariance

- The L-matrix theorem shows how a random orthogonal matrix (ROM) $\mathbf{R}_n$ is used, in conjunction with $\mathbf{L}_{mn}$, to generate a random sample $\mathbf{X}_{mn} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ as

$$\mathbf{x}_i = m^{\frac{1}{2}} \sigma_i \mathbf{y}_i + \mathbf{1}_m \mu_i$$

  Here μ_i and σ_i are the (known, target) mean and standard deviation of the ith variable

- Each random sample has *exactly* the target covariance matrix $\mathbf{V}_n$
- The sample characteristics depend on the choice of $\mathbf{R}_n$, and on the properties of the L-matrix

Skewness and Kurtosis of L-Matrices

- Since Mardia's measures of skewness and kurtosis are invariant under non-singular affine transformations, the skewness (kurtosis) of any ROM simulated sample is equal to the skewness (kurtosis) of the sample generated by the L-matrix alone
- It can be shown that

$$\begin{aligned} \tau_M(\mathbf{L}_{mn}) &= n\Big[(m-3)+(m-n)^{-1}\Big], \\ \kappa_M(\mathbf{L}_{mn}) &= n\Big[(m-2)+(m-n)^{-1}\Big]. \end{aligned}$$

- So, as in the MVN case, skewness and kurtosis are closely related. In fact, in ROM simulation $\tau_M = \kappa_M - n$
- The number of variables n is usually fixed, hence we can target either skewness or kurtosis by choosing the sample size m

Repeated Simulation

- Consider r samples $\mathbf{X}_{m_1 n}, \ldots, \mathbf{X}_{m_r n}$ for $i = 1 \ldots r$
- Let $m = \sum m_i$ and define $\mathbf{X}_{mn} = (\mathbf{X}'_{m_1 n}, \ldots, \mathbf{X}'_{m_r n})'$
- Then
$$\kappa_M(\mathbf{X}_{mn}) = mr^{-2} \sum \kappa_M(\mathbf{X}_{m_i n}) m_i^{-1}$$
- In particular if $m_1 = \cdots = m_r$ then simply
$$\kappa_M(\mathbf{X}_{mn}) = r^{-1} \sum \kappa_M(\mathbf{X}_{m_i n})$$
- Hence, with ROM simulations (of equal sample size and with the same target moments) the first four moments are all *unchanged* by repeated simulations

ROM Simulation Techniques

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Summary of Properties of ROM Simulation

- The rectangular orthogonal matrix $\mathbf{L}_{mn}$ plays a crucial role in ROM simulation
- $\mathbf{L}_{mn}$ is post-multiplied by products of many random orthogonal matrices (ROMS) in the matrix $\mathbf{R}_n$, to obtain a sample $\mathbf{Y}_{mn} = \mathbf{L}_{mn}\mathbf{R}_n\mathbf{A}_n$ with mean exactly $\mathbf{0}_n$ and covariance matrix exactly $\mathbf{V}_n = \mathbf{A}_n'\mathbf{A}_n$
- Then we apply an affine transformation to generate a sample with means and covariance matrix exactly as target
- The multivariate skewness and kurtosis of the sample is $n[(m-3)+(m-n)^{-1}]$ and $n[(m-2)+(m-n)^{-1}]$ respectively
- Assuming the number of variables n is fixed, we target skewness or kurtosis by choosing the sample size m. Repeating simulations does not alter the first four moments of the multivariate distribution.

Controlling Sample Characteristics

We now explain how the choice of random orthogonal matrices influences the *dynamic* characteristics of the sample, and the skewness and kurtosis of the *marginals*

Structure of L-matrix induces:

- Many zeros at beginning of the sample
- A cluster of volatility and correlation at end of the sample

The random orthogonal matrices (ROMs) in the product $\mathbf{R}_n$, where $\mathbf{Y}_{mn} = \mathbf{L}_{mn}\mathbf{R}_n\mathbf{A}_n$, may be choosen to:

- Disperse cluster $\Longrightarrow$ arbitrary permutation matrix
- Change location of cluster $\Longrightarrow$ cyclic permutation matrix
- Target marginal kurtosis $\Longrightarrow$ Hessenberg matrices
- Target marginal skewness $\Longrightarrow$ Sign matrices

Permutation Matrices

- A permutation matrix $\mathbf{P}_n$ of order n is a square orthogonal matrix which has exactly one entry 1 in every row and every column, and has entries 0 everywhere else.
- Permutes the columns of any matrix which it post-multiplies, e.g.

$$\mathbf{A}_{23}\mathbf{P}_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \end{pmatrix}$$

- Cyclic permuation matrices simply cycle the columns of a matrix (as in the example above)

Effect of Random Permutation Matrices

Sample paths generated using (a) only the L-matrix and then adding (b) a random cyclic permutation matrix or (c) a general random permutation matrix

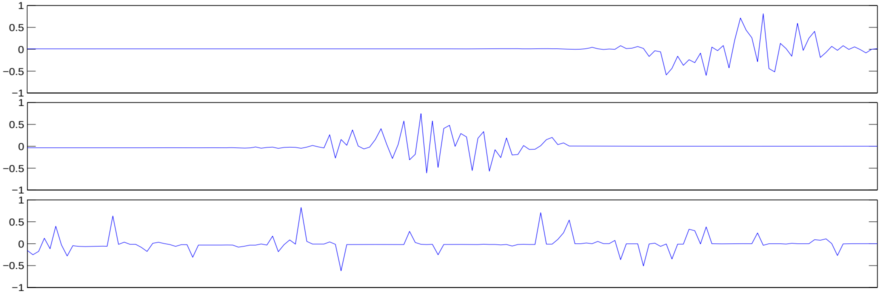


Figure: The simulations are based on a symmetric circulant correlation matrix of degree 25 but only the sample paths for the 10th random variable are shown

Hessenberg Matrices

- An orthogonal (upper) Hessenberg matrix $\mathbf{H}_n$, of degree n, has zero entries below the first subdiagonal and can be constructed as a product of $n-1$ Givens rotations via

$$\mathbf{H}_n = \mathbf{G}_1(\theta_1)\mathbf{G}_2(\theta_2)\ldots\mathbf{G}_{n-1}(\theta_{n-1})$$

- A Givens rotation $\mathbf{G}_j(\theta_j)$ is the identity matrix except for the 2×2 principal submatrix

$$\mathbf{G}_j[j, j+1; j, j+1] = \begin{pmatrix} cos(\theta_j) & -sin(\theta_j) \\ sin(\theta_j) & cos(\theta_j) \end{pmatrix}$$

- Choose θ_j randomly in the interval $[0, 2\pi)$ for $1 \leq j \leq n-1$

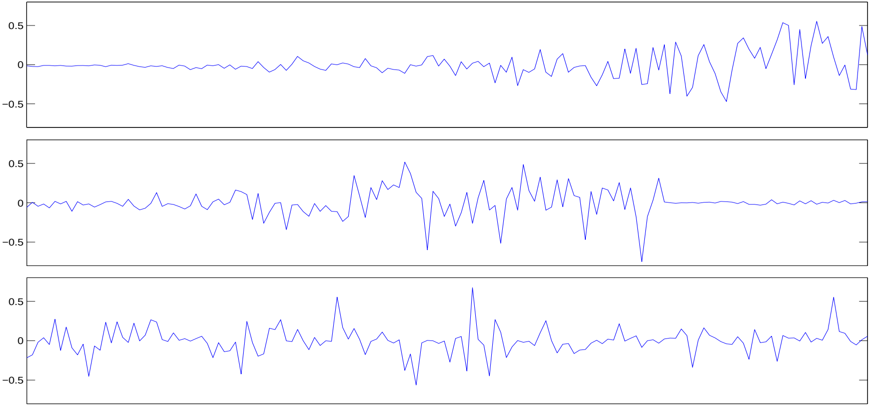
Example: Hessenberg Matrix of Degree 4

$$\mathbf{H}_4 = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & -s_2 & 0 \\ 0 & s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_3 & -s_3 \\ 0 & 0 & s_3 & c_3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & -s_1 c_2 & s_1 s_2 c_3 & -s_1 s_2 s_3 \\ s_1 & c_1 c_2 & -c_1 s_2 c_3 & c_1 s_2 s_3 \\ 0 & s_2 & c_2 c_3 & -c_2 s_3 \\ 0 & 0 & s_3 & c_3 \end{pmatrix}$$

Effect of Random Hessenberg Matrices

- Sample paths generated using (a) only the L-matrix and then adding (b) a random cyclic permutation matrix or (c) a general random permutation matrix, plus 500 additional Hessenberg matrices in each case



Random Sign Matrices

- We induce either positive or negative skewness into marginal distributions of ROM simulated samples using a random orthogonal matrix $\mathbf{B}_m$ of the form

$$\mathbf{B}_m = \text{diag}\left\{(-1)^{d_1}, \ldots, (-1)^{d_m}\right\}.$$

  where each d_i is randomly chosen to be either zero or one.

- We call such a matrix a random sign matrix. It will multiply the rows of any matrix it pre-multiplies by plus or minus one.

- We set $d_i \sim \mathrm{B}(1, p_i)$ for $1 \leq i \leq m$ and for some judicious choice of the probabilities $p_1, \ldots, p_m$

Effect of Random Sign Matrices

- Density plots where negative and positive skewness have been imposed using appropriate choice for $p_1, \ldots, p_m$.

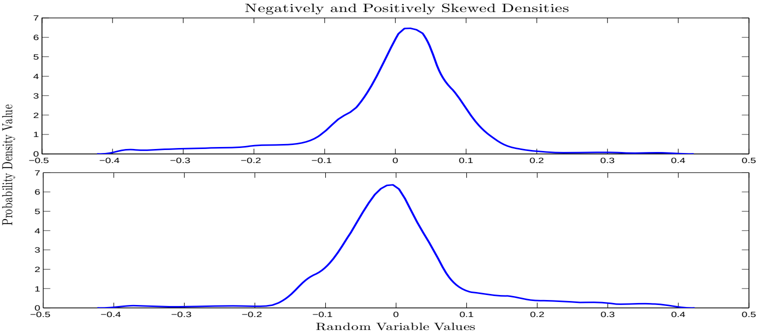


Figure: Plots show the 1st random variable generated from a symmetric circulant correlation matrix of order 5. Densities based on 10,000 simulated observations.

Parametric Perturbations

- This hybrid ROM-MC simulation technique adds a parametric random variable to the variable generated by the random orthogonal matrices
- In the ROM simulations $\mathbf{Y}_{mn} = \mathbf{L}_{mn}\mathbf{R}_n\mathbf{A}_n$ replace $\mathbf{L}_{mn}$ by $\tilde{\mathbf{L}}_{mn}$ constructed as follows:
- Take a parametric, zero mean, random sample $\mathbf{Z}_{mn}$ and apply Gram-Schmidt orthogonalization to the augmented matrix $(\mathbf{L}_{mn}, \mathbf{Z}_{mn})$ to obtain $(\hat{\mathbf{L}}_{mn}, \hat{\mathbf{Z}}_{mn})$
- Then $\hat{\mathbf{L}}_{mn} = \mathbf{L}_{mn}$ and $\mathbf{L}'_{mn}\hat{\mathbf{Z}}_{mn} = \mathbf{0}_n$
- Now, for some small ϵ, the matrix $\mathbf{L}_{mn}$ is replaced by the orthogonal matrix

$$\tilde{\mathbf{L}}_{mn} = \frac{1}{\sqrt{1+\epsilon^2}}\left(\mathbf{L}_{mn} + \epsilon\hat{\mathbf{Z}}_{mn}\right)$$

We can show that

$$\kappa_M(\tilde{\mathbf{L}}_{mn}) \approx \frac{\kappa_M(\mathbf{L}_{mn})}{(1+\epsilon^2)^2}$$

Distribution Targeting

- Targeting a given terminal distribution is the second hybrid ROM-MC simulation techniques introduced in the paper
- Recall

$$\mathbf{x}_i = m^{\frac{1}{2}} \sigma_i \mathbf{y}_i + \mathbf{1}_m \mu_i$$

- In distribution-targeted ROM simulations, instead of fixing μ_i for all simulations, we assume the μ_i are random
- Target a distribution for $(\mu_1, \ldots, \mu_n)$ keeping path standard deviations and correlations fixed, but changing the mean vector randomly in each simulation
- However, the MC simulation from the distribution of $(\mu_1, \ldots, \mu_n)$ introduces sampling error in distribution-targeted ROM simulations

Example Applications: (a) Portfolio VaR and (b) Corridor Options

Limitations of Standard VaR Methodolgies

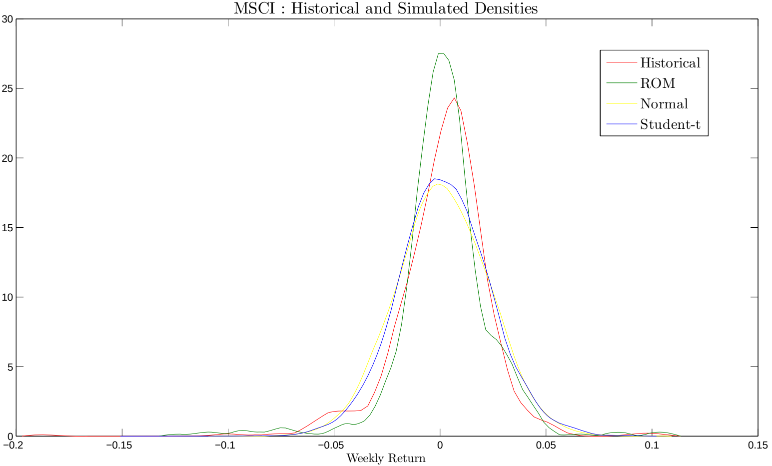
- Parametric VaR with an *analytic* solution is limited to elliptical and mixtures of elliptical distributions; yet these rarely capture the empirical characteristics of portfolio returns
- Hence Monte Carlo or historical simulation methods are common. However these have limitations:
  - MC simulation (a) requires parametric assumption and (b) has sampling error
  - Historical simulation (a) requires a very long in-sample period and (b) assumes history will repeat itself
- ROM simulation has features in common with both these methods:
  - Semiparametric (moment replication); virtually no sampling error
  - Replicates a potentially infinite number of samples that are consistent with moments of historical returns; does not require a long in-sample period

Example: Investing in MSCI World Indices

- Sample: weekly returns to 47 MSCI World country indices January 1995 to April 2009 (743 observations)
- Test accuracy of different VaR models using **1% and 5% weekly VaR** estimates for an **equally weighted portfolio** based on rolling **5-year** data window
- Unfortunately, (at least) 260 observations are required for historical simulation!
- ROM and MC simulations based on 10,000 simulations
- Use elliptical distributions for MC simulations, so that we can assess size of sampling error. In our experiment, this is between about 1.5% and about 2% of the VaR: it increases with significance level and with complexity of returns distribution

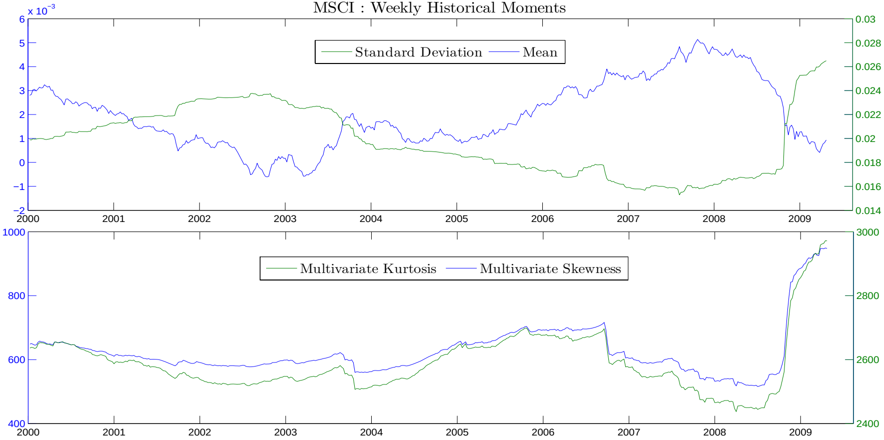
Distribution of Portfolio Returns

- Density of weekly returns to equally weighted portfolio based on entire data period from January 1995 to April 2009

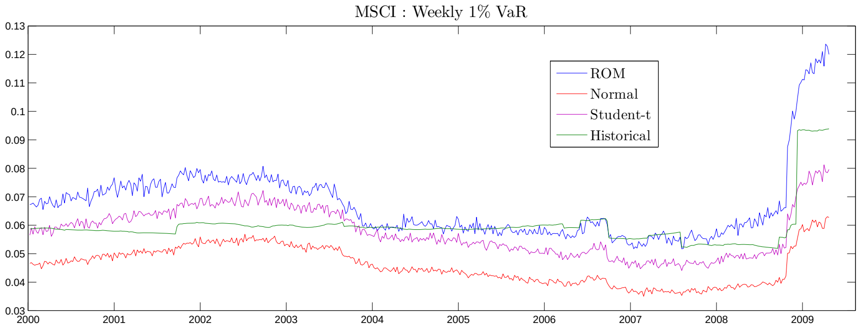


Evolution of Sample Moments

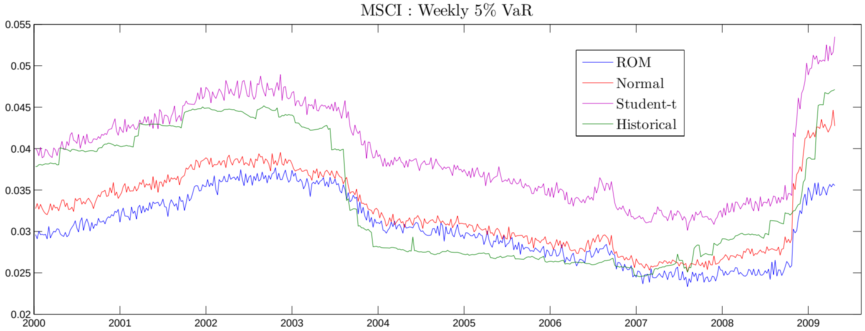
- Portfolio mean and standard devistion, and multivariate skewness and kurtosis, based on 5-year rolling window



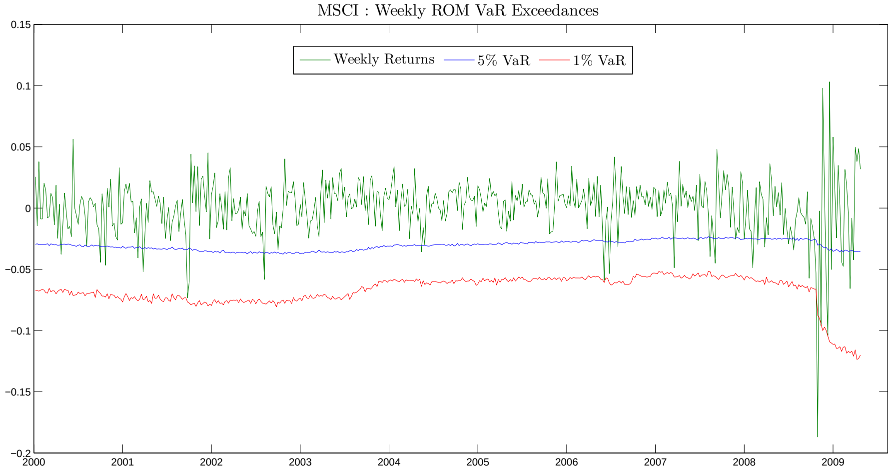
Comparison of 1% Weekly VaR Estimates



Comparison of 5% Weekly VaR Estimates



VaR Exceedences: ROM Simulation



Results of Coverage Tests for Model Specification

Coverage Test	Unconditional		Independence		Conditional	
VaR Level	1%	5%	1%	5%	1%	5%
ROM	2.16	3.73	0.02	1.08	2.18	4.81
Historical	0.01	0.60	0.10	2.93	0.11	3.53
Normal	18.71	2.41	5.63	1.58	24.34	3.99
Student t	2.88	0.81	2.07	3.91	4.96	4.72
5% Critical	3.84	3.84	3.84	3.84	5.99	5.99

Table: Coverage Test Statistics. Those shown in red reject the null hypothesis.

Example: Pricing Options with Path Dependent Pay-offs

- Under the GBM

$$\frac{dS_t}{S_t} = \sigma dW_t$$

we have

$$\log S_T \sim N(-\frac{1}{2}\sigma^2 T, \sigma^2 T) \tag{1}$$

- Distribution-targeted ROM simulation *only* makes the assumption (1)
- Experiment compares ROM simulated prices with GBM prices (which are, as usual, obtained by MC simulation) for a European digital corridor option. This has pay-off

$$\mathbf{1}_{\{B_L \leq S_t \leq B_U, 0 \leq t \leq T\}}$$

where S_t denotes the price of the underlying at time t, T is the maturity of the option and $[B_L, B_U]$ is the corridor

Results

ROM Simulations:

- Log normal price density targeted at 90 days
- One random cyclic permutation matrix to incorporate volatility clustering in log returns
- 250 random Hessenberg matrices $\Longrightarrow$ excess kurtosis in log returns.
- With volatilty clustering and excess kurtosis ROM prices should be lower than the GBM prices obtained using MC simulation

Digital Corridor $T = 90$	$[90, 110]$ ROM	$[90, 110]$ MC	$[75, 125]$ ROM	$[75, 125]$ MC
vol $= 20$ %	39.84%	43.38%	97.48%	97.56%
vol $= 40$ %	2.60%	2.88%	62.16%	64.08%

Table: ROM and MC simulation digital corridor prices, expressed as percentage of the pay-off, based on 5000 simulations, for different corridors and volatility levels.

Conclusion

- We introduced L-matrices and showed, using random orthogonal matrices (ROMs) that we can generate infinitely many random samples with exactly the target covariance matrix
- Exact covariance simulation can also be achieved by orthogonalizing MC simulations; but higher sample moments still have sampling error and parametric assumptions must be made
- ROM simulation is a new, semiparametric approach to simulation that targets higher sample moments exactly
- Sample charateristics (e.g. volatility clustering) are tailored using various well-known classes of random orthogonal matrices
- This, and various hybrid ROM-MC simulation techniques, appear to have useful applications to risk management, portfolio management and exotic option pricing (and to disciplines other than finance)
- Research is still in its early stages, but we hope for a considerable expansion of resources in the near future